UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
ADVANCED EXAM - Probability and Multivariate Distribution Theory
Friday, Jan. 14, 2011

Each problem is worth 20 point. 70 points are required to pass with at least 25 coming from question 1 and 2 and 25 coming from question 4 and 5.

1. (L^2 weak law of large numbers) Let $X_1, X_2, \cdots, X_n$ be uncorrelated random variables on the probability space $(\Omega, \mathcal{F}, P)$. Assume $\mu = E(X_i)$ and the variance of X is such that var$(X_i) \leq C < \infty$, for $i = 1, \cdots, n$. With $S_n = X_1 + \cdots + X_n$, *prove* that as $n \to \infty$, $S_n/n \to \mu$ in L^2 and in probability.

2. . (Central Limit Theorem) Let $X_1, \cdots, X_n$ be i.i.d. random variables on the probability space $(\Omega, \mathcal{F}, P)$. Assume $E(X_i) = \mu$, var$(X_i) = \sigma^2 \in (0, \infty)$, for any $i = 1, \cdots, n$. With $S_n = X_1 + \cdots + X_n$, *prove* that
$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \Rightarrow N(0,1)$$
where $N(0,1)$ is the standard normal distribution, and $\Rightarrow$ means to converge weakly or converge in distribution.

3. (a) Define the characteristic function and moment generating function (Laplace transform) of a random variable X. Do they always exist?

   (b) *Prove* that if $M(t)$ denotes the moment generating function of X then, under some conditions (specify what they are) for a positive integer k, $E(X^k) = \frac{d^k}{dt^k}M(t)$ evaluated at $t = 0$.

   A normal random variable with mean μ and variance σ^2 has moment generating function $M(t) = e^{t\mu + t^2\sigma^2/2}$. You can just use this below. No need to prove it.

   (c) Consider random variables $X_1, \ldots, X_n$ that are independent with X_i distributed normal with mean μ_i and variance σ_i^2. *Derive* the moment function of $S = \sum_{i=1}^{N} X_i$ (explain your steps) and hence show that S is normally distributed. As part of the result give what the mean and variance of S are.

4. Consider two random vectors $\mathbf{X}$ ($p \times 1$) and $\mathbf{Y}$ ($r \times 1$) and two matrices $\mathbf{A}$ ($a \times p$) and $\mathbf{B}$ ($b \times r$).

   (a) First define the covariance matrix for an individual random vector (e.g, $\mathbf{\Sigma}_X = Cov(\mathbf{X})$) and the covariance between two random vectors (e.g., $\mathbf{\Sigma}_{XY} = Cov(\mathbf{X}, \mathbf{Y})$) in terms of expected values.

(b) Show that if $\mathbf{X}$ and $\mathbf{Y}$ are independent then $Cov(\mathbf{X}, \mathbf{Y}) = \mathbf{0}$. Is the converse true? If yes, why? If no, give a counter example.

(c) *Derive* an expression for $Cov(\mathbf{AX})$ in terms of $\mathbf{A}$ and Σ_X.

(d) *Derive* an expression for $Cov(\mathbf{AX}, \mathbf{BY})$ with your answer given in terms of $\Sigma_X = Cov(\mathbf{X})$, $\Sigma_Y = Cov(\mathbf{Y})$ and $\Sigma_{XY} = Cov(\mathbf{X}, \mathbf{Y})$.

(e) Now suppose that $\mathbf{A}$ is a square $p \times p$ matrix and consider the quadratic form $Q = \mathbf{X}'\mathbf{AX}$ (where $'$ denotes transpose). Derive an expression $E(Q)$ with your answer given in terms of $\mathbf{A}$, $\boldsymbol{\mu} = E(\mathbf{X})$ and Σ_X and the trace operator. In doing this first define what the trace of a matrix is.

5. (Chi-square distribution)

(a) State the definition of a non-central chi-square distribution with d degrees of freedom and non-centrality parameter λ. Do this not by giving a density function but by explaining how the distribution arises as the distribution of a random variable C, formed as a function of a suitably defined normal random vector.

(b) Suppose now that $\mathbf{X}$ is distributed normal with mean $\boldsymbol{\mu}$ and covariance Σ. State a necessary and sufficient condition on the matrix $\mathbf{A}$ such that $\mathbf{X}'\mathbf{AX}$ is distributed as a non-central chi-square.

- Give the degrees of freedom and non-centrality parameter involved.

- Prove the sufficiency.