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Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.
2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.
3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.
4. State explicitly all results that you use in your proofs and verify that these results apply.
5. Please write your work and answers <u>clearly</u> in the blank space under each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.
2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.
3. If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space and let f be a measurable function such that

$$\varphi(t) \equiv \int e^{tf} d\mu$$

is finite for all $t \in I\!\!R$.

(a) Prove that for all $t \in I\!\!R$ the derivative $\varphi'(t)$ exists, and that $\int f d\mu$ exists and is given by

$$\int f d\mu = \varphi'(0) .$$

Hint: One possible proof uses the following inequality, which you can use without proof: for any real numbers h and x

$$\left| \frac{e^{hx} - 1}{h} \right| \leq e^{(1+|h|)|x|} < e^{(1+|h|)x} + e^{-(1+|h|)x} .$$

(b) Prove that for all $n \in I\!\!N$ and $t \in I\!\!R$ the n^{th} derivative $\varphi^{(n)}(t)$ exists, and that $\int f^n d\mu$ exists and is given by

$$\int f^n d\mu = \varphi^{(n)}(0) .$$

Hint: Use induction over n.

2. Let $\{F_n(z), n \in I\!N\}$ be a sequence of continuous functions mapping $I\!R$ into $I\!R$ with the following properties:

   (i) $F_n(z) \geq 0$ for all $z \in I\!R$ for all $n \in I\!N$.

   (ii) For any $\delta > 0$, $\lim_{n\to\infty} \int_{|z|>\delta} F_n(z)dz = 0$.

   (iii) $\int_{-\infty}^{\infty} F_n(z)dz = 1$ for all $n \in I\!N$.

   For any bounded, uniformly continuous function g mapping $I\!R$ into $I\!R$, define

$$K_n(x) = \int_{-\infty}^{\infty} F_n(x-y)g(y)\,dy\,.$$

   (a) Prove that

$$\lim_{n\to\infty} \sup_{x\in I\!R} |K_n(x) - g(x)| = 0\,.$$

   (b) Define $\psi_n(z) = \sqrt{n/2\pi} \cdot \exp(-nz^2/2)$. Prove that $\psi_n(z)$ satisfies the properties (i)–(iii).
   Hint: Use without proof the fact that $\int_{-\infty}^{\infty} \exp(-z^2/2)dz = \sqrt{2\pi}$.

3. Let f be a nonnegative Lebesgue integrable function on $[0, \infty)$. For $x \in [0, \infty)$ define

$$F(x) = \int_0^x f dm .$$

(a) Prove that F satisfies the properties required (via Carathéodory's procedure) to guaranteee the existence of a unique Borel measure μ_F on $[0, \infty)$ satisfying $\mu_F((a, b]) = F(b) - F(a)$ for all $0 \leq a < b < \infty$.

(b) Prove that $\mu_F \ll m$ and calculate the Radon-Nykodym derivative $\dfrac{d\mu_F}{dm}$.

4. Let $\mathcal{X}$ be a Banach space and let $U : \mathcal{X} \to I\!R$ be a linear map.

(a) Show that the following are equivalent

(i) U is continuous.

(ii) U is continuous at 0.

(iii) U is bounded.

(b) Give the definition of the norm of U, $\|U\|$.

(c) Let $L(\mathcal{X}, I\!R)$ denote the space of all bounded, linear maps U from $\mathcal{X}$ into $I\!R$. It is easily verified that $L(\mathcal{X}, I\!R)$ is a normed vector space with norm $\|U\|$. Prove that the normed vector space $L(\mathcal{X}, I\!R)$ is complete and is therefore a Banach space.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose that $f \in L^1(\mu) \cap L^2(\mu)$.

   (a) Prove that $f \in L^p(\mu)$ for $1 \leq p \leq 2$.

   (b) Prove that $\lim_{p \to 1+} \|f\|_p = \|f\|_1$.

   Hints: Consider the set $A = \{x \in X : |f(x)| \geq 1\}$ and its complement A^c. Use the Dominated Convergence Theorem for part (b).

6. For any measurable function f defined on $(0, \infty)$ define the function $g(s)$ for $s > 0$ by

$$g(s) = \int_0^\infty e^{-st} f(t)\, dt\,.$$

The function $g(s)$ is called the *Laplace transform of* f, provided the integral is finite for all $s > 0$.

(a) Prove that for $f \in L^2((0,\infty))$ the function $g(s)$ is finite for any $s > 0$.

(b) Prove the formula

$$\int_0^\infty e^{-st} t^{-1/2} dt = s^{-1/2}\sqrt{\pi}.$$

Hint: Use without proof the fact that $\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$.

(c) Using part (b), prove that

$$|g(s)|^2 \leq \sqrt{\pi} s^{-1/2} \int_0^\infty |f(t)|^2 e^{-st} t^{1/2} dt.$$

(d) Using parts (b) and (c), prove that $\|g\|_{L^2} \leq \pi \|f\|_{L^2}$ and thus that the Laplace transform maps $L^2((0,\infty))$ into itself.

7. The set $\left\{ \frac{1}{\sqrt{2\pi}} e^{inx}, n \in \mathbb{Z} \right\}$ is an orthonormal basis of the Hilbert space $L^2([-\pi, \pi])$. For $f \in L^2([-\pi, \pi])$ and $n \in \mathbb{Z}$ define

$$c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx$$

and for $k \in \mathbb{N}$ define

$$S_k = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}, |n| \leq k} c_n e^{inx}.$$

(a) In what sense does $S_k \to f$ as $k \to \infty$? Explain your answer.

(b) Using part (a), prove that for any real numbers a and b satisfying $-\pi \leq a < b \leq \pi$

$$\int_a^b f(x) \, dx \; = \; \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{2\pi}} \int_a^b c_n e^{inx} \, dx \, .$$

(c) Use part (b) to prove the formula

$$\frac{\pi^2}{8} \; = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \sum_{k \in \mathbb{N}} \frac{1}{(2k-1)^2}.$$

Hints: Choose $f(x) = x$, $a = 0$, and $b = \pi$, and evaluate c_n.

8. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{f_n, n \in N\}$ be a sequence of nonnegative integrable functions. Suppose there exists an integrable function f such that

   (i) $\lim_{n\to\infty} f_n(x) = f(x)$ almost everywhere.

   (ii) $\lim_{n\to\infty} \int_X f_n d\mu = \int f d\mu$.

   Given E any measurable subset of X, prove that

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu \,.$$

   Hints: Use without proof the fact that a sequence of real numbers converges to $x \in I\!R$ if every subsequence has a subsubsequence converging to x. Also use Fatou's Lemma for $f_n 1_E$ and $f_n 1_{E^c}$ for appropriate n.