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Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.
2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.
3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.
4. State explicitly all results that you use in your proofs and verify that these results apply.
5. Please write your work and answers clearly in the blank space under each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.
2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.
3. If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$

1. Let $A \subseteq \mathbb{R}$ be an arbitrary subset of the real line that is not necessarily Lebesgue measurable, and let $m^*(A)$ denote the Lebesgue outer measure of A. Prove that there exists a Borel set $B \subseteq \mathbb{R}$ such that $A \subseteq B$ and $m(B) = m^*(A)$.

2. Let $(X, \mathcal{M}, \mu)$ be a *finite* measure space and let $\{h_n, n \in \mathbb{N}\}$ be a sequence of nonnegative, Borel-measurable functions satisfying $h_n \to 0$ in L^1 as $n \to \infty$.

   (a) Prove that $\sqrt{h_n} \to 0$ in L^1 as $n \to \infty$. (**Hint.** For each n split X into the set where $h_n \geq \delta$ and the set where $h_n < \delta$.)

   (b) Give an example to show that h_n^2 need not converge to 0 in L^1 as $n \to \infty$.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n, n \in \mathbb{N}\}$ be a sequence of integrable functions that converges in measure to another integrable function $f \in L^1(\mu)$. Define $g(x) = \sup_{n\in\mathbb{N}} |f_n(x)|$ for $x \in X$, and assume that g is integrable. Prove that f_n converges to f in L^1.

4. Let X be a space, and let $X = A_1 \cup A_2 \cup \ldots \cup A_n$ be a finite partition of X into n disjoint, nonempty sets $A_1, A_2, \ldots, A_n$. Define $\mathcal{M}$ to be the σ-algebra generated by $A_1, A_2, \ldots, A_n$ (i.e., the collection of sets that are the unions of some, none, or all of the A_j). Let μ be a finite, positive measure on $\mathcal{M}$ such that $0 < \mu(A_j) < \infty$ for all $j = 1, 2, \ldots, n$, and let ν be a finite, positive measure on $\mathcal{B}$. Prove that ν is absolutely continuous with respect to μ and that for all $j = 1, 2, \ldots, n$ and all $x \in A_j$

$$\frac{d\nu}{d\mu}(x) = \frac{\nu(A_j)}{\mu(A_j)}.$$

5. Let $(X, \mathcal{M}, \mu)$ be a measure space, $\{g_n, n \in \mathbb{N}\}$ a sequence of measurable functions mapping X into $\mathbb{R}$, g a measurable function mapping X into $\mathbb{R}$, and p a real number satisfying $1 \leq p < \infty$.

(a) Define the concepts that $g_n \to g$ in measure and that $\|g_n - g\|_p \to 0$ (convergence in $L^p(\mu)$).

(b) Prove that if $\|g_n - g\|_p \to 0$, then $g_n \to g$ in measure.

(c) Assume that there exists a nonnegative $h \in L^p(\mu)$ such that $|g_n(x)| \leq h(x)$ for all $x \in X$. Prove that if $g_n \to g$ in measure, then $\|g_n - g\|_p \to 0$. In order to do this, use without proof the following formula, valid for any measurable function f on X:

$$\int_X |f|^p \, d\mu = p \int_0^\infty \alpha^{p-1} \, \sigma_f(\alpha) \, d\alpha,$$

where $\sigma_f(\alpha) = \mu(\{x \in X : |f(x)| > \alpha\})$. (**Hint.** Prove that $|g(x)| \leq h(x)$ a.e. by using a property of a subsequence of g_n.)

6. Let f and g be nonnegative, Borel-measurable, Lebesgue-integrable functions on $\mathbb{R}$. For $x \in \mathbb{R}$ the convolution of f and g is defined by

$$f * g(x) = \int_{\mathbb{R}} f(x-t)g(t)dm(t).$$

(a) Prove that $f * g(x) = g * f(x)$ for all $x \in \mathbb{R}$.

(b) Prove that

$$\int_{\mathbb{R}} f * g(x)\, dm(x) = \int_{\mathbb{R}} f(x)dm(x) \cdot \int_{\mathbb{R}} g(x)dm(x),$$

and conclude that $f * g \in L^1(m)$. (**Hint.** Use without proof that fact that if a function h is Borel-measurable on $\mathbb{R}$, then the function $H(x,t) = h(x-t)$ is Borel-measurable on $\mathbb{R} \times \mathbb{R}$).

7. Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let F be a subset of H. We denote by $\overline{F}$ the smallest closed subspace of $\mathcal{H}$ containing F. We define the set

$$F^{\perp} = \{u \in \mathcal{H} : \langle u, x \rangle = 0 \text{ for all } x \in F\}.$$

(a) Prove that $F^{\perp}$ is a closed subspace of $\mathcal{H}$.

(b) Prove that $F \subset (F^{\perp})^{\perp}$ and that $\overline{F}^{\perp} \subset F^{\perp}$.

(c) For any closed subspace K of $\mathcal{H}$ the following is true: $(K^{\perp})^{\perp} = K$. Using this fact (which you need not prove) and part (b), prove that $(F^{\perp})^{\perp} = \overline{F}$.

8. (a) State Hölder's Inequality.

(b) State Minkowski's Inequality.

(c) Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a measure space and let α, β, and γ be real numbers satisfying $1 < \alpha < \infty$, $1 < \beta < \infty$, $1 < \gamma < \infty$, and

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1.$$

Use Hölder's Inequality to prove that if $f \in L^{\alpha}(\mu)$, $g \in L^{\beta}(\mu)$, and $h \in L^{\gamma}(\mu)$, then

$$\int_{\mathcal{X}} |f\, g\, h|\, d\mu \le \|f\|_{\alpha}\, \|g\|_{\beta}\, \|h\|_{\gamma} < \infty.$$