NAME:

Advanced Probability Qualifying Examination
Department of Mathematics and Statistics
University of Massachusetts

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Instructions

1. This exam consists of six (6) problems (each of equal weight 20). You need to solve 5 out of 6 problems and your grade will be evaluated using the five problems you choose (or the best five out of six problems if you decide to solve all the problems).

2. In order to pass this exam, it is enough that you solve essentially correctly at least three (3) problems and that you have an overall score of at least 65%.

3. State explicitly all results that you use in your proofs and verify that these results apply.

4. Please write your work and answers clearly in the blank space under each question.

5. The last page is empty and can be used if you need more space.

1. (a) You flip a fair coin. If the coin lands on "Tail" you lose \$1 while if it lands on "Heads" you generate a random number U (i.e., a uniform random variable on $[0, 1]$) and you gain is equal to \$ $2U$. If Y denotes your gain (in \$) find the distribution function $F_Y(x)$ of Y.

   (b) Let X be the random variable with distribution function

$$F_X(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{2}\sqrt{x/2} & 1 \leq x < 2 \\ 1 & 2 \leq x \end{cases} .$$

   Compute $P(\frac{3}{2} < x \leq 2)$ and $E[X]$.

2. Let $\{X_n\}$, $n = 1, 2, 3, \cdots$, be a sequence of independent and identically distributed random variable, each of them exponentially distributed with parameter 1, i.e.,

$$P(X_n > x) = e^{-x}\,, \quad x > 0\,.$$

   (a) For any $\alpha > 0$ compute

$$P\left(\frac{X_n}{\log n} > \alpha \;\text{ i.o.}\right)\,.$$

   (b) Deduce from (a) that

$$\limsup_{n\to\infty} \frac{X_n}{\log n} = 1 \;\text{ almost surely}\,.$$

3. Let $\{X_n\}$, $n = 1, 2, 3, \cdots$, and X be random variables on a probability space $(\Omega, \mathcal{F}, P)$.

   (a) Give precise definitions of the following convergence concepts, as $n \to \infty$:
   (i) $X_n \to X$ almost surely; (ii) $X_n \to X$ in probability;
   (iii) $X_n \to X$ in distribution. (iv) $X_n \to X$ in L^2.

   (b) Prove that if X_n converges to X in L^2 then X_n converges to X in probability, as $n \to \infty$.

   (c) Give an example showing that almost sure convergence does not imply convergence in L_2.

4. Let $\{X_n\}$, $n = 0, 1, 2, 3 \cdots$, be a stochastic process taking value in a countable state space S.

   (a) Give the definition of "X_n is a Markov process".

   (b) Show that X_n is a Markov process if and only if the future and the past are conditionally independent given the present, that is, if and only if we have

   $$P(X_0 = i_0, \cdots, X_{n-1} = i_{n-1}, X_{n+1} = i_{n+1} \cdots X_{n+k} = i_{n+k} \,|\, X_n = i_n)$$
   $$= P(X_0 = i_0, \cdots, X_{n-1} = i_{n-1} \,|\, X_n = i_n) P(X_{n+1} = i_{n+1} \cdots X_{n+k} = i_{n+k} \,|\, X_n = i_n)$$

   for any n and k and any states $i_0, \cdots i_{n+k}$.

   (c) Suppose $S = \{1, 2, 3, 4, 5, 6\}$ and X_n is a Markov chain with state space S. Is it true or not that

   $$P(X_2 = 6 \,|\, X_1 \in \{3, 4\}, X_0 = 2) = P(X_2 = 6 \,|\, X_1 \in \{3, 4\})?$$

   You need to justify your answer.

5. Cars arrive at a gas station which consists of a single pump according to a Poisson process with rate λ. A car can enter the station only if fewer than four cars are present. The amount of time to serve a car is exponentially distributed with expected service time $1/\mu$.

   (a) Define a continuous Markov chain and specify its transition rates (its infinitesimal generator) and compute its stationary distribution.

   (b) What is the long-run average number of cars in the station?

   (c) What is the long-run fraction of time the pump is occupied?

   (d) What is the long-run fraction of cars that cannot enter the station?

6. Consider the following algorithm to compute the number e. At the j^{th} step of the algorithm, generate random numbers $U_1, U_2, \cdots$ until the descending order is broker, i.e. set

$$N = \inf\{n \geq 1, U_1 > U_2 > U_3 > \cdots > U_{n-1} < U_n\}$$

If N is even then set $Y_j = 1$ otherwise set $Y_j = 0$.

Show that

$$\lim_{k \to \infty} \frac{Y_1 + Y_2 + \cdots Y_k}{k} = 1 - e^{-1} \quad \text{almost surely}$$

Hint: Compute first $P(N > n)$.