UMass Amherst Algebra Advanced Exam
Friday January 16, 2014, 10AM – 1PM.

Instructions: To pass the exam it is sufficient to solve five problems including a least one problem from each of the three parts. Show all your work and justify your answers carefully.

1. Group theory and representation theory

Q1. Suppose G is a group with 60 elements with the further property that the order of the center of G is divisible by 4. Show that G is abelian.

Q2. Let G be a finite group and let N be a normal subgroup of index a prime p. Let C be a conjugacy class of G which is contained in N. Show that either C is still a conjugacy class in N or else splits into p conjugacy classes of equal size.

Q3. Let $\mathbb{F}_q$ be the finite field of order q. Let G be the group of invertible affine linear maps

$$g\colon \mathbb{F}_q \to \mathbb{F}_q, \quad x \mapsto ax + b.$$

where $a \in \mathbb{F}_q \setminus \{0\}$ and $b \in \mathbb{F}_q$, with the group law being composition of maps.

(a) List the conjugacy classes of G.
(b) Determine the abelianization of G.
(c) Compute the dimensions of the irreducible complex representations of G.
(d) By its definition G acts on the set $\mathbb{F}_q$. Let ρ be the associated permutation representation. Then we can write $\rho = \rho_1 \oplus \rho'$ where ρ_1 is the trivial representation. Compute the character of ρ' and show that ρ' is irreducible.

2. Commutative Algebra

Q4. Let R be a principal ideal domain. Let A, B, C be finitely generated R-modules such that $A \oplus B \cong A \oplus C$. Show that $B \cong C$.

Q5. Let R be a reduced commutative ring (that is, R has no nonzero nilpotent elements) which has exactly one prime ideal. Prove that R is a field.

Q6. Let R denote the ring $\mathbb{Z}[\sqrt{-5}]$ and p the ideal $(2, 1 + \sqrt{-5}) \subset R$.

(a) Show that p is a prime ideal.

(b) Using the norm

$$N\colon R \to \mathbb{Z}, \quad N(a + b\sqrt{-5}) = a^2 + 5b^2$$

or otherwise, show that p is *not* a principal ideal.

(c) Let $S = R \setminus p$ and consider the localization $S^{-1}R$ of R at p. Show that the ideal $S^{-1}p \subset S^{-1}R$ is principal.

3. Field theory and Galois theory

Q7. Let $\alpha = \sqrt{4 + \sqrt{7}}$.

(a) Find the minimal polynomial f of α over $\mathbb{Q}$. (You should show carefully that f is irreducible over $\mathbb{Q}$.)

(b) Let K be the splitting field of f over $\mathbb{Q}$. Determine the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ of K over $\mathbb{Q}$.

Q8. Let $F \subset K$ be a Galois extension with Galois group isomorphic to the alternating group A_4. Let $\alpha \in K$ be an element. Determine the possible values of the degree of the minimal polynomial of α over F.

Q9. Let $f = (x^3 - 2)(x^2 - 3)$. Let K be the splitting field of f over $\mathbb{Q}$. Determine the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ of K over $\mathbb{Q}$.