**DEPARTMENT OF MATHEMATICS AND STATISTICS**
**UNIVERSITY OF MASSACHUSETTS, AMHERST**

# ADVANCED EXAM — ALGEBRA

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**Passing Standard:** It is sufficient to do FIVE problems correctly, including at least ONE FROM EACH of the four parts.

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## Part I.

1. Let $P$ be a Sylow $p$-subgroup of a finite group $G$. For any normal subgroup $N$ of $G$, show that $N \cap P$ is a Sylow $p$-subgroup of $N$.

2. (a) Let $G$ be a subgroup of the symmetric group $S_n$. Show that either all elements of $G$ are even permutations, or exactly half of them are.
(b) Show that the alternating group $A_4$ is the unique subgroup of $S_4$ of order 12 (by definition, $A_4$ is the set of all even permutations in $S_4$).

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## Part II.

1. Let $A$ be a commutative ring with identity. If the polynomial ring $A[x]$ is a PID, show that $A$ must be a field.

2. (a) Show that the ascending chain condition holds for *principal ideals* of any UFD.
(b) Give an example of a UFD for which the ascending chain condition does not hold for *all ideals*. NOTE: you need to show that the ring which you claim to be a UFD is in fact a UFD.

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## Part III.

1. (a) Let $A$ be a PID. Suppose $M$ is a finitely generated module over $A$. If $M \otimes_A M = 0$, show that $M = 0$.
(b) Show that $\mathbf{Q}/\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z} = 0$.
(c) What do Parts (a) and (b) together imply about $\mathbf{Q}/\mathbf{Z}$?

2. Let $R$ be a commutative ring with 1, and let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be a short exact sequence of $R$-modules.
(a) Prove or disprove: $M$ is a finitely generated $R$-module if and only if $L$ and $N$ are.
(b) Prove or disprove: $M$ is torsion-free if and only if $L$ and $N$ are.

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## Part IV.

1. Let $\mathbf{F}_q$ be a finite field with $q > 2$ elements. For any non-zero integer $k$ relatively prime to $q - 1$, show that

$$\sum_{\alpha \in \mathbf{F}_q} \alpha^k = 0.$$

2. Let $K/k$ be a finite field extension. Suppose $L_1$ and $L_2$ are intermediate subfields of $K/k$ such that $L_1 \cap L_2 = k$.

(a) Suppose $L_1/k$ is Galois. Show that the minimal polynomial of any $\alpha \in L_1$ over $k$ is irreducible over $L_2$.

(b) Give an example that shows Part (a) is false if $L_1/k$ is *not* Galois.