M523Honors: Introduction to Modern Analysis

Homework

Spring 2012

Note: Do not turn in yet the Special Projects nor problems followed by an asterisk $*$. I'll announce when these are due.

Assignment 1. Due Thursday February 16th 2012.

Extra Problem 1*: Show that $(\mathbb{C}, +, \cdot)$ as defined in class is a field.

From Section 1.1: 1, 3, 4, 5, 6a), 7, 10, 11, 12$*$

From Section 1.2: 1, 2, 3, 5a), 5d), 6a), 6b), 6c), 7a), 7b), 7d), 9b), 10.

From Section 1.3: 1, 3, 5, 7, 8, 9*.

Assignment 2. Due Thursday February 23 2012.

From Section 1.4: 1, 4, 7, 8, 9, 10a)c) 11a)

(In 10c) f^2 means $[f(x)]^2$ and **not** $f \circ f$)

From Section 2.1: 1, 2, 3, 4, 5, 7

Assignment 3. Due Thursday March 8th 2012.

From Section 2.2: 1, 2, 3, 4, 5(modified) , 7, 8, 9.

Problem 5(modified) Prove theorem 2.26 in the special case when $a_n = 1$. That is prove $\frac{1}{b_n} \to \frac{1}{b}$ under the same hypothesis. I will post a handout for the general case later.

From Section 2.4: 1, 2, 3, 5, 6, 7, 9, 13.

Special Project I: Do Project #1 at the end of Chapter 2.

Assignment 4. Due Thursday March 15th 2012.

From Section 1.3 Problem 9

From Section 2.5: 1, 3(modified), 4, 5, 7, 8.

Problem 3(modified). *Suppose a set S of real numbers is bounded and let η be a lower bound for S. Show that η is the greatest lower bound of S if and only if for every $\varepsilon > 0$ there is an element of S in the interval $[\eta, \eta + \varepsilon]$*

Assignment 5. Due Thursday March 29th 2012.

Extra Problem 2* Prove the existence of *greatest lower bounds* just as we proved the existence of *least upper bound* in Theorem 2.5.1

From Section 2.6: 1, 2, 3, 4, 6, 8, 9, 10, 11, 13.

Special Project II: Do Problem 14 of Section 2.4.

Special Project III: Do Project #5 at the end of Chapter 2 (page 70)

Assignment 6. Due Thursday 4/12/2012.

From Section 3.1 : 2, 4, 5, 7, 8a)b), 9, 11.

Assignment 7. Due Thursday 4/19/2012.

From Section 3.2 : 1, 3, 4, 5, 7, 10, 11.

From Section 3.3: 1, 2, 3, 4, 5, 7, 12, 14, 15.

Assignment 8. Due Tuesday 5/1/2012.

From Section 3.5: 1, 2, 3, 7, 8.

From Section 5.1: 2, 7, 8, 12.

From Section 5.2: 1, 2a), 6.

Hints For 5.1 #7: for each $n \geq 1$ choose an x in $[0,1]$ such that $n\,x = 1$. Call that x, x_n and compute $f_n(x_n)$.

For 5.1 #8: for each $n \geq 1$ choose an x in $[0,1]$ such that $\dfrac{x}{n} = 1$. Call that x, x_n and compute $f_n(x_n)$.

For 5.1 #12: Given $\varepsilon > 0$, write $|f_n(x_n) - f(x_0)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)|$ and find $N = N(\varepsilon)$ so that (a) the first term on the r.h.s of the inequality is less than $\varepsilon/2$ thanks to the *uniform convergence* of f_n to f; (b) the second term on the r.h.s of the inequality is less than $\varepsilon/2$ thanks to the *continuity* of f

For 5.2 #2*a*): first prove that the sequence of functions $f_n(x) = (x + \frac{1}{n})^2$ converges uniformly to the function $f(x) = x$ on $[0,1]$ as n goes to infinity. Then use Theorem 5.2.2 to compute.

For 5.2 #6: Denote by f the limiting function and write $|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)|$.
First note that since the convergence is uniform on $[0,1]$, f must be continuous (why?) and hence bounded (why?). Second, prove that there exists N (think of $\varepsilon = 1$) such that for all $n \geq N$ the first term on the right hand side is less than 1. Third, note that each of the remaining functions f_n, $1 \leq n \leq N-1$ is continuous and bounded on $[0,1]$ (and there are only a finite $N-1$, a finite number of them).

Finally, put all the ingredients together to conclude!

Special Projects (**Due no later than 1PM on Friday 5/04/12**)

Special Project I: Do Project #1 at the end of Chapter 2 (page 68)

Special Project II: Do Problem 14 of Section 2.4

Special Project III: Do Project #5 at the end of Chapter 2 (page 70)

Special Project IV: Do Problem 13 of Section 3.2

Special Project V:

Let f be a continuous function on $\mathbb{R}$ such that the improper integral $\int_{-\infty}^{\infty} f(x)\,dx < \infty$.

Let f_n be a sequence of continuous functions defined on $\mathbb{R}$ such that f_n converge uniformly to f on every finite, closed interval $[a,b]$ of $\mathbb{R}$.

Suppose that there exists a **continuous** function $g : \mathbb{R} \to \mathbb{R}$ such that:

(i) $g(x) \geq 0$
(ii) the improper integral $\int_{-\infty}^{\infty} g(x)\,dx < \infty$,
(iii) for all $n \geq 1$ and all $x \in \mathbb{R}$ we have that $|f_n(x)| \leq g(x)$ and also $|f(x)| \leq g(x)$.

(a) Prove that each of the improper integrals $\int_{-\infty}^{\infty} f_n(x)\,dx < \infty$

(b) Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

Hint for b) The improper integral $\int_{-\infty}^{\infty} g(x)\,dx = \lim_{M\to\infty} \int_{-M}^{M} g(x)\,dx$ and since $\int_{-\infty}^{\infty} g(x)\,dx < \infty$ then the limit in M of the sequence $\int_{-M}^{M} g(x)\,dx$ of real numbers (why are each of these finite?) exists.

Hence given $\varepsilon > 0$ there exists an $M_0 = M_0(\varepsilon) > 0$ such that

$$\left| \int_{-\infty}^{\infty} g(x)\,dx - \int_{-M}^{M} g(x)\,dx \right| = \left| \int_{|x|>M} g(x)\,dx \right| \leq \varepsilon \qquad M \geq M_0.$$

To prove part for b), you need to consider

$$\left| \int_{-\infty}^{\infty} f_n(x)\,dx - \int_{-\infty}^{\infty} f(x)\,dx \right| = \left| \int_{-\infty}^{\infty} \Big(f_n(x) - f(x) \Big)\,dx \right| \qquad (\dagger)$$

Next, rewrite the r.h.s in (†) as the sum of two integrals, one over the set $|x| \leq M$ and the other over the set $|x| > M$ and use triangle inequality to bound (†) by

$$\left|\int_{|x|\leq M}\Big(f_n(x)\ -\ f(x)\Big)\,dx\right|\ +\ \left|\int_{|x|>M}\Big(f_n(x)\ -\ f(x)\Big)\,dx\right|.$$

Use uniform convergence over the set $|x| \leq M$. For the integral over the set, $|x| > M$, use triangle inequality, the hypothesis (iii) and part (b) to bound each term by integrals over $|x| > M$ of $g(x)$.

Put all the pieces together to conclude!.