Lecture 15: Difference equations, gambling and random walks

In this lecture we discuss how to solve *linear difference equations* and give several applications.

First order homogeneous equation: Think of the time taking integer values $n = 0, 1, 2, \cdots$ and q_n describing the state of some system at time n. We consider an equation of the form

$$\text{First order homogeneous} \quad aq_n + bq_{n-1} = 0$$

where q_n, $n = 0, 1, 2, 3, \cdots$ are unknown and a and b are fixed constants. This equation is called a *first order homogeneous* equation. To solve it we rewrite it as

$$q_n = \left(\frac{-b}{a}\right) q_{n-1} = \alpha q_{n-1}$$

with $\alpha = -b/a$. This is easy to solve recursively

$$q_n = \alpha q_{n-1} = \alpha(\alpha q_{n-2}) \; = \; \alpha^2 q_{n-1} \; = \; \cdots \; = \; \alpha^n q_0$$

So if we are given q_0, e.g. the state of the system at time 0, then the state of the system at time n is given by $q_n = \alpha^n q_0$, i.e. this is a model for exponential growth or decay.

To summarize

$$\text{The general solution of } aq_n + bq_{n-1} = 0 \text{ is } q_n = C\left(-b/a\right)^n$$

Interest rate: A bank account gives an interest rate of 5% compounded monthly. If you invest \$1000, how much money do you have after 10 years? Since the interest is paid monthly we set

$$q_n = \text{ amount of money after } n \text{ months}$$

and since we get one twelfth of 5% every month we have

$$q_n \; = \; \left(1 + \frac{.05}{12}\right) q_{n-1} \; = \; \left(1 + \frac{1}{240}\right) q_{n-1} \; = \; \left(\frac{241}{240}\right) q_{n-1}$$

and so after 5 year we have with $q_0 = 1000$

$$q_{60} = \left(\frac{241}{240}\right)^{60} 1000 = 1283.35$$

First order inhomogeneous equation: Let us consider an equation of the form

$$\boxed{\text{First order inhomogeneous} \quad aq_n + bq_{n-1} = c_n,}$$

where c_n is a *given* sequence and q_n is unknown. For example we may take

$$c_n = c\,, \quad c_n = cn\,, \quad c_n = c\alpha^n\,.$$

This equation is called *inhomogeneous* because of the term c_n. The following simple fact is useful to solve such equations

Linearity principle: *Suppose $\hat{q}_n$ be a solution of the inhomogeneous $aq_n + bq_{n-1} = c_n$ and $\tilde{q}_n$ be a solution of the homogeneous equation $aq_n + bq_{n-1} = 0$ then $q_n + \tilde{q}_n$ is a solution of the inhomegenous equation $aq_n + bq_{n-1} = c_n$. Indeed we have*

$$\begin{aligned} a\hat{q}_n + b\hat{q}_{n-1} &= c_n \\ a\tilde{q}_n + b\tilde{q}_{n-1} &= 0 \end{aligned}$$

and thus adding the two equations give

$$a(\hat{q}_n + \tilde{q}_n) + b(\hat{q}_{n-1} + \tilde{q}_{n-1}) = c_n \tag{1}$$

To find the general solution of a first order homogeneous equation we need

- Find *one particular* solution of the inhomogeneous equation.
- Find the general solution of the homogeneous equation. This solution has a free constant in it which we then determine using for example the value of q_0.
- The general solution of the inhomogeneous equation is the sum of the particular solution of the inhomogeneous equation and general solution of the homogeneous equation.

Example: Solve

$$aq_n + bq_{n-1} = c$$

i.e., the inhomegenous term is $c_n = c$ i.e. constant. We look for a particular solution, and after some head scratching we try $q_n = d$ to be constant and find

$$ad + bd = c\,, \qquad \text{or } d = \frac{c}{a+b}$$

The general solution is then

$$q_n \;=\; C(-b/a)^n + \frac{c}{a+b}\,.$$

Example: Solve

$$2q_n - q_{n-1} = 2^n\,, \quad q_0 = 3$$

The solution of the homogenous is $q^n = C(1/2)^n$. To find a particular solution of the inhomogeneous problem we try an exponential function $q_n = D2^n$ with a constant D to be determined. Plugging into the equation we find

$$2D2^n - D2^{n-1} = 2^n$$

or after dividing by 2^{n-1}

$$4D - D = 2 \text{ or } D = \frac{2}{3}\,.$$

So the general solution is

$$q_n \;=\; C\left(\frac{1}{2}\right)^n + \frac{2}{3}2^n\,.$$

and the initial condition gives $q_0 = 3 = C + \frac{2}{3}$ and so

$$q_n \;=\; \frac{7}{3}\left(\frac{1}{2}\right)^n + \frac{2}{3}2^n\,.$$

More interest rate: A bank account gives an interest rate of 5% compounded monthly. If you invest invest initially \$1000, and add \$10 every month. How much money do you have after 10 years? Since the interest is paid monthly we set

$$q_n = \text{ amount of money after } n \text{ months}$$

and we have the equation for q_n

$$q_n = \left(1 + \frac{.05}{12}\right) q_{n-1} + 10 \;=\; \left(\frac{241}{240}\right) q_{n-1} + 10$$

For the particular solution we try $q_n = d$ and find

$$d = \frac{241}{240}d + 10$$

i.e., $d = -2400$. The general solution is then

$$q_n = C\left(\frac{241}{240}\right)^n - 2400$$

and $q_0 = 1000$ gives

$$q_n \;=\; 3400\left(\frac{241}{240}\right)^n - 2400$$

and so $q_60 = 1963.41$

Second order homogeneous equation: We consider an equation where q_n depends on both q_{n-1} and q_{n-2}:

Second order homogeneous $\;aq_n + bq_{n-1} + cq_{n-2} = 0\,.$

It is easy to see that we are given both q_0 and q_1 we can determine q_2, q_3, and so on.

Linearity Principle: *It is easy to verify that if $\tilde{q}_n$ and $\hat{q}_n$ are two solutions of the second order homogeneous equation. Then $C_1\tilde{q}_n + C_2\hat{q}_n$ is also a solution for any constant C_1, C_2.*

To find the general solution we get inspired by the homogeneous first order equation and look for solutions of the form

$$q_n = x^n$$

If we plug this into the equation we find

$$a\alpha^n + b\alpha^{n-1} + c\alpha^{n-2}$$

and dividing by α^{n-2} give

$$ax^2 + bx + c \;=\; 0$$

We find (in general) two roots x_1 and x_2 and the general solution has the form

$$q_n = C_1 x_1^n + C_2 x_2^n$$

Example: Solve the second order equation $3q_n + 5q_{n-1} - 2q_{n-2} = 0$, with $q_0 = 2$ and $q_1 = -3$. If we try $q_n = x^n$ we obtain the quadratic equation

$$3x^2 + 5x - 2 = 0 \text{ or } x = 1/3, -2$$

and so the general solution is

$$q_n = C_1 \left(\frac{1}{3}\right)^n + C_2(-2)^n$$

The initial conditions give $2 = C_1 + C_2$ and $-3 = C_1(1/3) + C_2(-2)$. This gives $C_1 = 3/7$ and $C_2 = 11/7$ and so

$$q_n = \frac{3}{7}\left(\frac{1}{3}\right)^n + \frac{11}{7}(-2)^n$$

Second order inhomogeneous equation: We consider an equation of the form

Second order homogeneous $aq_n + bq_{n-1} + cq_{n-2} = d_n$.

where q_n is unknown and d_n is a fixed sequence. As for first order equations we can solve such equations by

1. Solve the homogeneous equation $aq_n + bq_{n-1} + cq_{n-2} = 0$.
2. Find a particular solution of the inhomogeneous equation.
3. Write the general solution as the sum of the particular inhomogeneous equation plus the general solution of the homogeneous equation.

Application: Random walk and Gambler's ruin. Consider the set of integers $\mathbf{N} = \{0, 1, 2, 3, 4, \cdots\}$. Imagine a walker moving along a line, every unit of time he makes a step and each step he makes is exactly one unit so that his position at any time is some integer. The rule for the motion of the walker starting at n are

$$\begin{aligned} \text{If in position } n \qquad & \text{move to } n+1 \text{ with probability } p \\ & \text{move to } n-1 \text{ with probability } q \\ & \text{stay at } n \text{ with probability } r \end{aligned} \tag{2}$$

with

$$p + q + r = 1$$

Instead a walker on a line you can think of a gambler at a casino making bets of $1 at certain game (say red on roulette). He start with a fortune of $n. With probability p he doubles his bet, that he increase his fortune by $1, with probability q he looses and his fortune decreases by \$1, and with probability r he gets his bet back and his fortune is unchanged.

As we have seen in previous lectures, in many such games the odds of winning are very close to 1 with p typically around .51. Using our second order difference equations we will show that even though the odds are only very slightly in favor of the casino, this enough to ensure that in the long run, the casino will makes lots of money and the gambler not so much. We also investigate what is the better strategy for a gambler, play small amounts of money (be cautious) or play big amounts of money (be bold). We shall see that being bold is the better strategy if odds are not in your favor (i.e. in casino), while if the odds are in your favor the better strategy is to play small amounts of money.

In order to make the previous problem more precise we imagine the following situation.

- You starting fortune if \$j.
- In every game you bet \$1.
- Your decide to play until you either loose it all (i.e., your fortune is 0) or you fortune reaches \$N and you then quit.
- What is the probability to win?

We denote by A_j the event that you win starting with \$j.

$$\begin{aligned} q_j &= P(A_j) = \text{Probability to win starting with } j \\ &= \text{Probability to reach } N \text{ before reaching 0 starting from } j \end{aligned} \tag{3}$$

To compute q_j we use the formula for conditional probability and condition on what happens at the first game, win, lose, or tie. For every game we have

$$P(\text{win}) = p, \quad P(\text{lose}) = q, \quad , P(\text{tie}) = r$$

We have

$$\begin{aligned} q_j &= P(A_j) \\ &= P(A_j|\text{win})P(\text{win}) + P(A_j|\text{lose})P(\text{lose}) + P(A_j|\text{tie})P(\text{tie}) \\ &= \qquad q_{j+1} \times p \qquad + \qquad q_j \times q \qquad + \qquad q_{j-1} \times r \end{aligned} \tag{4}$$

since if we win the first game, our fortune is then $j+1$, and the the probability that we then win is simply q_{j+1}, and son on....

Using that $p+q+r=1$ we can rewrite this as the second order equation

$$pq_{j+1} - (p+q)q_j + qq_{j-1} \; = \; 0$$

With $q_j = x^j$ we find the quadratic equation

$$px^2 - (p+q)x + q = 0$$

with solutions

$$x = \frac{-p \pm \sqrt{(p+q)^2 - 4pq}}{2p} = \frac{-p \pm \sqrt{p^2 + q^2 - 2pq}}{2p} = \frac{-p \pm \sqrt{(p-q)^2}}{2p} = \begin{cases} 1 \\ q/p \end{cases}$$

and so the general solution is given by

$$q_n = C_1 1^n + C_2 \left(\frac{q}{p}\right)^n$$

To determine the constants C_1 and C_2 we use that

$$q_0 = 0\,, \quad \text{and } q_N = 1\,.$$

which follow from the definition of q_j as the probability to win (i.e. reaching a fortune of N) starting with a fortune of j. We find

$$0 = C_1 + C_2\,, \qquad 1 = C_1 + C_2 \left(\frac{q}{p}\right)^N$$

which gives

$$C_1 = -C_2 = \left(1 - \left(\frac{q}{p}\right)^N\right)^{-1}\,,$$

and so we find

$$\boxed{\text{Gambler's ruin probabilities } \; q_n = \frac{1 - (q/p)^n}{1 - (q/p)^N}}$$

To give an idea on how this function look like let us take

$$q = .51, p = 0.49 r = 0,$$

see figure 1 and let us pick N=100. That is we start with a fortune of j and wish to reach a fortune of 100. We have for example.

$$q_{10} = 0.0091, \quad q_{50} = 0.1191, q_{75} = 0.3560, \quad q_{83} = 0.4973.$$

That is starting with \$10 the probability to win \$100 before losing all you money is only about one in hundred. If you start half-way to your goal, that is with \$50, the probability to win is still a not so great 11 in one hundred and you reach a fifty-fifty chance to win only if you start with \$83.

Bold or cautious? Using the formula for the gambler's ruin we can investigate whether there is a better gambling strategy than betting \$1 repeatedly. For example if we start with \$10 and our goal is to reach \$100 we choose between

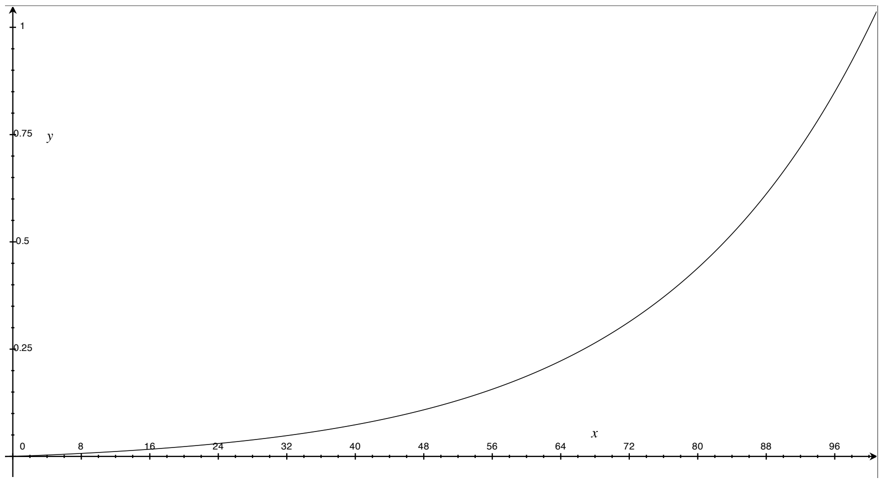

Figure 1: Gambler's ruin probabilities

- Play in \$1 bets?
- Play in \$10 bets?

We find

$$\text{Probability to win \$100 in \$1 bets starting with \$10 is } q_{10} = \frac{1-(51/49)^{10}}{1-(51/49)^{100}} = 0.0091$$

while for the other case we use the same formula with $N = 10$ and $j = 1$ since we need to make a net total of 9 wins

$$\text{Probability to win \$100 in \$10 bets starting with \$10 is } q_{11} = \frac{1-(51/49)}{1-(51/49)^{10}} = 0.0829$$

that is your chance to win is about 8 in hundred, about nine time better than by playing in \$1 increments. Based on such arguments it seems clear the best strategy is to be bold if the odds of the game are not in your favor.

If on the contrary the odds are in your favor, even so slightly, say q=.49, and p=.51 then the opposite is true: you should play cautiously. For example with these probabilities and in the same situation as before, starting with \$ 10 and with a \$100 goal we find

$$\text{Probability to win \$100 in \$1 bets starting with \$10 is } q_{10} = \frac{1-(49/51)^{10}}{1-(49/51)^{100}} = 0.3358$$

while for the other case we use the same formula with $N = 10$ and $j = 1$ since we need to make a net total of 9 wins

$$\text{Probability to win \$100 in \$10 bets starting with \$10 is } q_{11} = \frac{1-(49/51)}{1-(49/51)^{10}} = 0.1189.$$

In summary we have

> If the odds are in your favor be cautious but if the odds are against you be bold!

Exercise 1: Solve the following difference equation

1. $2q_n - 5q_{n-1} = 0$, $q_0 = 2$
2. $2q_n - 5q_{n-1} = 3$, $q_0 = 3$
3. $2q_{n+1} - 7q_n + 3q_{n-1} \; = \; 0, \quad q_0 = 1, q_1 = 2$
4. $2q_{n+1} - 7q_n + 3q_{n-1} \; = \; 2 + 2^n, \quad q_0 = 1, q_1 = 2$

Exercise 2: Your retirement account has a fixed rate of 8% per year *paid yearly.* You start saving for retirement at age 30 with a target retirement age of 65 and $0 in your saving account. Set-up and solve a suitable first order difference equations to answer the following questions.

1. Suppose you set aside $500 every month. How much money will you have for your retirement?
2. You want to retire with $500'000. How much should you save every month?
3. Assuming that your salary is going to increase 5 % per year during your life you also decide you contribution will increase by 5% every year. If your starting contribution is $500 every month how much money will you have saved at retirement age?
4. Assuming again that your contribution is increasing by 5% every year, what should your starting contribution be if if you want to reach $500'000 by retirement age?

Exercise 3: Your mortgage is a 30 year fixed rate mortgage at an (fixed) annual rate of 4% compounded monthly.

1. If you borrow $150'000 today, what is the total amount of money will you pay back to the bank during the next 30 years?

2. You decide that you can make a down-payment of \$15000 and that \$1250 is the maximal monthly payment you are willing to commit to. How much house can you buy?

Exercise 4: At a certain casino game if you bet \$x you will loose your \$x with probability .505 (so your fortune will idecrease by x) and win \$2x with probability \$.495 (so your fortune will increase by x). You walk into the casino with \$25 dollars with the goal to get \$200. Compute the probability for you to succeed if you use the following strategies

1. You make repeated \$5 bets until you either win \$200 or you are wiped out.
2. You make repeated \$25 bets until you either win \$200 or you are wiped out.
3. You keep betting all your available money, i.e., first bet \$25 then bet \$50 if you win the first bet, and so on.

Exercise 5: The following game is proposed to you. You have to perform a random walk on the integer between 0 and N where you go one step left or one step right with probability 1/2. But there is a twist: each time you make a step there is a probability 1/10 that you win the game immediately. You win this game if either you win immediately at some step or if you reach N first. You loose if you reach 0 first.

1. Argue that the probability to win starting at j satisfies the second order equation

$$q_j = \frac{1}{10} + .45 q_{j+1} + .45 q_{j-1}\,, \qquad q_0 = 0, q_N = 1$$

2. Compute q_j. What happens to q_j when $N \to \infty$.

Exercise 6: Martingale strategy. At a certain casino game if you bet \$x you will loose your \$x with probability q and win \$2x with probability p.

Your great aunt just left you with an inheritance of 2560000 (that is 2^8 times thousand dollars) and you decided to move to Las Vegas. Being a gambler, a slacker, and a nerd you decide to use the following strategy to stay in Vegas for as long as possible.

- At the beginning of each month you go to the casino and bet \$10000. If you win you made a profit of \$10'000, you quit and live for a month on your \$10000 gain until you return next month.

- If you loose your first bet, you double your bet and stake \$20000. If you win this second bet, you made a profit of \$10'000 (why?) and again you quit and live for a month on your \$10000 gain.

- If you loose your first two bets, then you double your bet to \$40'000, etc....

1. Show that every month you either win \$10000 of go bankrupt.

2. Find the probability that you go bankrupt in any single given month. What if $p = 1/2$?

3. Find the probability that you go bankrupt exactly in your n^{th} visit to the casino? What if $p = 1/2$?

4. Find the expected number of times you visit the casino until you go bankrupt. What if $p = 1/2$?
   Hint: To compute $\sum_{n=1}^{\infty} n x^{n-1}$ compute first $\sum_{n=0}^{\infty} x^n$ and then differentiate with respect to x.

5. Find the expected return on your investment, that is compute the expected amount of money you made in casino. What if $p = 1/2$?